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Reflection anomaly in $0+1$ dimensions

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Abstract. The (0+1)-dimensional field theory of Dirac fermions coupled to an external Abelian gauge field is quantized in the path integral formalism. The imaginary part of the effective action, which has many features in common with the $(2+1)$-dimensional effective action, is the sum of a $\mathbb{Z}_2$-violating Chern–Simons term plus a certain $\mathbb{Z}_2$-preserving and nonanalytic term related to the integer part of the Chern–Simons action. The effective coupling constant of the theory can be any real number depending on the regularization scheme. The physical origin of the $\mathbb{Z}_2$ anomaly in odd-dimensional gauge theories is analysed from a nonperturbative point of view.

1. Introduction

One of the most important features of the quantization of Dirac fermions in $(2+1)$-dimensional gauge theories is the appearance of the parity anomaly [1–5]. Properly speaking, a parity transformation of the gauge field $A_\mu(t,\boldsymbol{x})$ in $2+1$ dimensions is defined by means of the transformation $A_\mu(t,\boldsymbol{x}) \longrightarrow (A_0(t,\pm x_1,\mp x_2), \pm A_1(t,\pm x_1,\mp x_2), \mp A_2(t,\pm x_1,\mp x_2))$. However, very often, 'parity transformation' is used in the literature for a different $\mathbb{Z}_2$ symmetry [6–8]: the reflection symmetry, defined by $A_\mu(t,\boldsymbol{x}) \longrightarrow -A_\mu(-t,-\boldsymbol{x})$. Therefore, although we should distinguish between parity and reflection symmetry, both are usually referred to as 'parity symmetry'. Obviously, the term becomes inadequate for $0+1$ dimensions and in what follows, we shall distinguish clearly between these two symmetries.

The effective action generated by the integration of the massless fermionic degrees of freedom in $(2+1)$-dimensional gauge theories defined over a three-dimensional compact manifold $\mathcal{M}$ is given by [1–8]

$$\operatorname{Im}\Gamma(A) = k_{\text{eff}}(-\mathrm{i}S_{\text{CS}}(A) + 2\pi h[A]) \tag{1}$$

where $S_{\text{CS}}(A)$ is the $(2+1)$-dimensional Chern–Simons term

$$S_{\text{CS}}(A) \equiv \frac{\mathrm{i}}{4\pi}\int_{\mathcal{M}} \operatorname{Tr}(A\wedge \mathrm{d}A + \tfrac{2}{3}A\wedge A\wedge A)$$

and $h[A]$ is a certain nonanalytic function of the gauge field $A:\mathcal{M}\longrightarrow$ Lie algebra of $U(1)$ or $SU(N)$. However, the literature [1–6], and especially the recent literature [7–9, 11], about the quantization of $(2+1)$-dimensional Dirac fermions contains heterogeneous and sometimes contradictory information. The major controversy is related to the value of k_{eff}, which may depend on the regularization method prescribed for its calculation. In the non-Abelian case, some authors obtain [1–3] $k_{\text{eff}} = \pm\frac{1}{2}$, whereas $k_{\text{eff}} = p/2$ is obtained in [8], where p may be any integer number. Moreover, in [7] it is argued that k_{eff} may be any real number depending on the parameters of the regularization. The controversy is similar in the Abelian case [4, 9]. As a consequence, some authors propose the absence of parity (and reflection) anomaly in the theory [7–9].

On one hand, the Chern–Simons action is antisymmetric under a reflection transformation and, under a gauge transformation $g : \mathcal{M} \longrightarrow SU(N)$ of the gauge field, $A \longrightarrow A^g = g^{-1}Ag - \mathrm{i}g^{-1}\,\mathrm{d}g$,

$$S_{\mathrm{CS}}(A) \to S_{\mathrm{CS}}(A^g) = S_{\mathrm{CS}}(A) - 2n\pi\mathrm{i} \tag{2}$$

where n is the winding number of g. On the other hand, although the explicit form of $h[A]$ is unknown, some properties of this term are well established [2, 7]. It is integer and jumps ± 1 when, for varying A, one eigenvalue of the Dirac operator vanishes. Moreover, the gauge and reflection transformations of $h[A]$ are given by

$$h[A^g] = h[A] + n \tag{3}$$

and

$$h[-A] = -h[A] = h[A] \pm 2|h[A]| \tag{4}$$

respectively. Therefore, using (2)–(4) we see that $\exp \operatorname{Im} \Gamma(A)$ is gauge invariant, but not reflection invariant.

The Chern–Simons term may be obtained by means of perturbation theory, but the term $h[A]$ is not generated radiatively. The existence and physical origin of a term $h[A]$ verifying the above properties has been shown by Redlich in [2] by using the Atiyah–Singer theorem in $3+1$ dimensions. The necessity of the presence of this term in the imaginary part of the effective non-Abelian action has been also argued in [7] in order to assure the gauge invariance of $\exp \Gamma(A)$, but the explicit form and physical origin of the term $h[A]$ is unknown. The purpose of this paper is to calculate explicitly the $(0+1)$-dimensional term $h[A]$, compare the ambiguities of the $(0+1)$-dimensional coefficient k_{eff} with those of the $(2+1)$-dimensional one and analyse the physical origin of the reflection anomaly in odd dimensions. The aim of this study is to shed some light on the possible realizations of the more physically interesting $(2+1)$-dimensional $h[A]$ term and on the physical origin of the discrete anomalies.

In the next section, the effective action generated by the integration of Dirac fermions interacting with an Abelian gauge field and defined in a $(0+1)$-dimensional compact manifold is calculated exactly by means of a standard Pauli–Villars regularization. The nonanalytic term $h[A]$ is obtained explicitly. In section 3, the ambiguities in k_{eff} are analysed by means of a generalization of the Pauli–Villars scheme. In section 4 a physical explanation about the origin of the reflection anomaly in $0+1$ dimensions (in odd dimensions in general) is proposed. Some conclusions and a few comments are postponed to section 5.

2. The model: Dirac fermions in the circle

Consider a $(0+1)$-dimensional system [12] of massless fermions interacting with an Abelian gauge field at finite time T (finite temperature). The classical interaction action of this system is given by

$$S(A, \psi^*, \psi) \equiv \int_{-T}^{T} \psi^*(t) \left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)\right) \psi(t)\,\mathrm{d}t \tag{5}$$

where $A(t) : S^T \longrightarrow \mathbb{R}$ is the gauge field and $\psi(t) : S^T \longrightarrow \mathbb{C}$ is the fermion field†. The gauge field verifies periodic boundary conditions, $A(t+2T) = A(t)$ whereas the fermion field verifies antiperiodic boundary conditions, $\psi(t+2T) = -\psi(t)$.

† By S^T we mean the circle of length $2T$.

The quantum effects of the fermions on the gauge field are encoded in the effective action $\Gamma(A)$. It is defined by means of the integration of the fermionic degrees of freedom and normalized to 1 for $A = 0$,

$$\mathrm{e}^{\Gamma(A)} \equiv \frac{\int \delta\psi^* \delta\psi \, \mathrm{e}^{-S(A,\psi^*,\psi)}}{\int \delta\psi^* \delta\psi \, \mathrm{e}^{-S(0,\psi^*,\psi)}} = \frac{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)]}{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}]} \tag{6}$$

where $\int \delta\psi^* \delta\psi$ means that the integrand $\mathrm{e}^{-S(A,\psi^*,\psi)}$ must be integrated over all the fields ψ^* and ψ verifying the antiperiodic boundary conditions mentioned above.

Therefore, in order to calculate the effective action $\Gamma(A)$, we need to know the eigenvalues of the $(0+1)$-dimensional Dirac operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$. It is well known that the eigenvalues of this operator acting on fermions defined over the circle S^T are $\lambda_k(\epsilon) \equiv \pi(k + \epsilon + \frac{1}{2})/T$, $k \in Z$, where ϵ is the $(0+1)$-dimensional Chern–Simons action,

$$\epsilon \equiv \frac{1}{2\pi} \int_{-T}^{T} A(t) \, \mathrm{d}t.$$

Inserting $\lambda_k(\epsilon)$ into the right-hand side of (6) we verify that this expression is divergent. Therefore, we must introduce a regularization scheme. In order to explore possible ambiguities in k_{eff} we must consider a very general regularization scheme, but this is postponed to the next section. Now, in order to better understand the origin of the nonanalytic term $h[A]$, we introduce a standard Pauli–Villars regularization,

$$\begin{aligned}
\mathrm{e}^{\Gamma(A)} &= \lim_{M\to\infty} \frac{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)]}{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}]} \frac{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{i}M]}{\det[-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t) + \mathrm{i}M]} \\
&= \lim_{M\to\infty} \prod_{k\in Z} \left[\frac{k + \epsilon + \frac{1}{2}}{k + \frac{1}{2}}\right] \left[\frac{k + \epsilon + \frac{1}{2} + \mathrm{i}M}{k + \frac{1}{2} + \mathrm{i}M}\right]^{-1} \\
&= \lim_{M\to\infty} \left[(2\epsilon + 1) \prod_{k=1}^{\infty} \left(1 + \frac{\epsilon}{k + \frac{1}{2}}\right)\left(1 - \frac{\epsilon}{k - \frac{1}{2}}\right)\right] \\
&\quad \times \left[\prod_{k=1}^{\infty} \left(1 + \frac{\epsilon}{k + \frac{1}{2} + \mathrm{i}M}\right)\left(1 - \frac{\epsilon}{k - \frac{1}{2} - \mathrm{i}M}\right)\right]^{-1}.
\end{aligned} \tag{7}$$

Dune *et al* [12] have computed the second quotient of determinants in the first line of the above equation for arbitrary M. The limits $M \to 0$ and $M \to \infty$ of that calculation imply that the first bracket in the right-hand side of (7) equals $\cos(-\pi\epsilon)$, whereas the second one equals $\mathrm{e}^{-\mathrm{i}\pi\epsilon} + \mathrm{O}(M^{-1})$. Therefore,

$$\mathrm{e}^{\Gamma(A)} = |\cos(\pi\epsilon)|(-1)^{\mathrm{Int}(\epsilon + \frac{1}{2})}(-1)^{-\epsilon} \tag{8}$$

and the imaginary part of the effective action reads

$$\mathrm{Im}\,\Gamma(A) = \pi(-\epsilon + \mathrm{Int}(\epsilon + \tfrac{1}{2})). \tag{9}$$

We can identify clearly here the explicit $(0+1)$-dimensional form of the nonanalytic term $h[A]$. It is given by

$$h[A] = \mathrm{Int}(\epsilon + \tfrac{1}{2}). \tag{10}$$

It has been generated by the determinant of the physical Dirac fermion and obviously, it cannot be found by means of a perturbative expansion in ϵ. Under a gauge transformation $\mathrm{e}^{\mathrm{i}\phi(t)} : S^T \longrightarrow U(1)$ of winding number n, $\epsilon \longrightarrow \epsilon + n$ and under a reflection transformation, $\epsilon \longrightarrow -\epsilon$. Therefore, the term $\mathrm{Int}(\epsilon + \frac{1}{2})$ verifies all the properties mentioned in the introduction for the $(2+1)$-dimensional case: it is integer, transforms under a gauge transformation in the form (3), under a parity transformation in the form (4) and jumps one unity when, for varying ϵ, one eigenvalue $k + \epsilon + \frac{1}{2}$ vanishes. The other term of the right-hand side of the above equation is just the Chern–Simons action multiplied by $-\pi$ and was generated by the Pauli–Villars determinant. It can be easily verified that this term is just what we obtain in perturbation theory. Therefore, as well as in $2+1$ dimensions, the radiative corrections only pick up the Chern–Simons term of $\mathrm{Im}\,\Gamma(A)$. Nevertheless, the imaginary part (9) of the $(0+1)$-dimensional effective action agrees with the $(2+1)$-dimensional non-Abelian one if we change $-\epsilon + \mathrm{Int}(\epsilon + \frac{1}{2})$ by its $(2+1)$-dimensional partner $-(\mathrm{i}/2\pi)S_{\mathrm{CS}}(A) + h[A]$ ($k_{\mathrm{eff}} = \frac{1}{2}$ in the non-Abelian case for $(2+1)$-dimensional Pauli–Villars regularizations).

3. Ambiguities in the reflection anomaly

In order to analyse the ambiguities in the $(0+1)$-dimensional coefficient k_{eff}, we consider now the regularization method introduced in [7]. This scheme has shown the ambiguities in the $(2+1)$-dimensional coefficient k_{eff}, generalizing the results obtained by other methods [1–6, 8–9, 11]. It is a gauge invariant generalization of the Pauli–Villars scheme that contains not only pseudoscalar couplings between the fermions and the gauge field, but also scalar couplings. This is achieved by means of high derivatives. In $0+1$ dimensions it reads,

$$\mathrm{e}^{\Gamma(A)} = \lim_{M\to\infty} \prod_{j=0}^{N} \frac{\det^{s_j}\left[d_A + \mathrm{i}\lambda_j \frac{d_A^2}{M}\left(1 + \frac{d_A^2}{M^2}\right)^{n_j} + \mathrm{i}\mu_j M\right]}{\det^{s_j}\left[d_0 + \mathrm{i}\lambda_j \frac{d_0^2}{M}\left(1 + \frac{d_0^2}{M^2}\right)^{n_j} + \mathrm{i}\mu_j M\right]} \tag{11}$$

where we have denoted

$$d_A = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t).$$

$s_0 = 1$, $s_j = \pm 1$ for $j = 1, 2, \ldots, N$, $2n_j$ are integer numbers and λ_j and μ_j are real numbers with $\mu_0 = 0$. In order to assure that (11) is a true regularization, we must find the Pauli–Villars conditions that make $\Gamma(A)$ finite for finite M. All the determinants involved in the regularization (11) are formally gauge invariant, therefore, as in standard Pauli–Villars regularizations, finiteness of every couple of quotients in the above formula is enough to guarantee gauge invariance. It is assured by choosing N odd and $s_j = (-1)^j$, $j = 0, 1, \ldots, N$. If we denote by $\Lambda_j(k + \epsilon + \frac{1}{2})$ the eigenvalues of the operator $d_A + \mathrm{i}\lambda_j(1 + d_A^2/M^2)^{n_j} d_A^2/M + \mathrm{i}\mu_j M$,

$$\Lambda_j(k) \equiv \frac{\pi}{T}\left[k + \mathrm{i}\frac{\lambda_j k^2}{M}\left(1 + \frac{k^2}{M^2}\right)^{n_j} + \mathrm{i}\mu_j M\right] \tag{12}$$

and each consecutive couple of quotients of determinants in (11) may be written

$$\begin{aligned} &\frac{\det[d_A + \mathrm{i}\lambda_j(1 + d_A^2/M^2)^{n_j} d_A^2/M + \mathrm{i}\mu_j M]}{\det[d_0 + \mathrm{i}\lambda_j(1 + d_0^2/M^2)^{n_j} d_0^2/M + \mathrm{i}\mu_j M]} \\ &\qquad \times \frac{\det[d_0 + \mathrm{i}\lambda_{j+1}(1 + d_0^2/M^2)^{n_{j+1}} d_0^2/M + \mathrm{i}\mu_{j+1} M]}{\det[d_A + \mathrm{i}\lambda_{j+1}(1 + d_A^2/M^2)^{n_{j+1}} d_A^2/M + \mathrm{i}\mu_{j+1} M]} \\ &\quad = \prod_{k\in Z} \frac{\Lambda_j(k + \epsilon + \frac{1}{2})}{\Lambda_j(k + \frac{1}{2})} \frac{\Lambda_{j+1}(k + \frac{1}{2})}{\Lambda_{j+1}(k + \epsilon + \frac{1}{2})} \\ &\quad = \prod_{k\in Z} \frac{\Lambda_j(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))}{\Lambda_j(k + \frac{1}{2})} \frac{\Lambda_{j+1}(k + \frac{1}{2})}{\Lambda_{j+1}(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))} \end{aligned}$$

$$\equiv \lim_{L\to\infty} \prod_{k=-L}^{L} \frac{\Lambda_j(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))}{\Lambda_j(k + \frac{1}{2})} \frac{\Lambda_{j+1}(k + \frac{1}{2})}{\Lambda_{j+1}(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))}. \tag{13}$$

In the last equality we have split $\epsilon + \frac{1}{2}$ into its integer and fractional parts, $\epsilon + \frac{1}{2} = \mathrm{Int}(\epsilon + \frac{1}{2}) + \mathrm{Frac}(\epsilon + \frac{1}{2})$. Therefore, the contribution of every one of these couple of quotients to the imaginary part of the effective action $\Gamma(A)$ is given by

$$\begin{aligned} \mathrm{Im}\,\log & \left\{ \frac{\det[d_A + \mathrm{i}\lambda_j(1 + d_A^2/M^2)^{n_j} d_A^2/M + \mathrm{i}\mu_j M]}{\det[d_0 + \mathrm{i}\lambda_j(1 + d_0^2/M^2)^{n_j} d_0^2/M + \mathrm{i}\mu_j M]} \right. \\ & \qquad \left. \times \frac{\det[d_0 + \mathrm{i}\lambda_{j+1}(1 + d_0^2/M^2)^{n_{j+1}} d_0^2/M + \mathrm{i}\mu_{j+1} M]}{\det[d_A + \mathrm{i}\lambda_{j+1}(1 + d_A^2/M^2)^{n_{j+1}} d_A^2/M + \mathrm{i}\mu_{j+1} M]} \right\} \\ & = \lim_{L\to\infty} [\Gamma_j(L) - \Gamma_{j+1}(L)] \end{aligned}$$

where

$$\Gamma_j(L) = \sum_{k=-L}^{L} \left[\tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))}{\mathrm{Re}\,\Lambda_j(k + \mathrm{Frac}(\epsilon + \frac{1}{2}))} \right) - \tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(k + \frac{1}{2})}{\mathrm{Re}\,\Lambda_j(k + \frac{1}{2})} \right) \right] \tag{14}$$

and

$$\frac{\mathrm{Im}\,\Lambda_j(k)}{\mathrm{Re}\,\Lambda_j(k)} = \frac{\lambda_j(k/M)^2(1 + (k/M)^2)^{n_j} + \mu_j}{k/M}.$$

For large M, we can substitute the discrete variable k/M by a continuum variable x and the summation in (14) by M times an integral in x,

$$\begin{aligned} \Gamma_j(L) = M & \left\{ \int_{\frac{-L+\mathrm{Frac}(\epsilon+\frac{1}{2})}{M}}^{\frac{-1+\mathrm{Frac}(\epsilon+\frac{1}{2})}{M}} + \int_{\frac{\mathrm{Frac}(\epsilon+\frac{1}{2})}{M}}^{\frac{L+\mathrm{Frac}(\epsilon+\frac{1}{2})}{M}} - \int_{\frac{-L+\frac{1}{2}}{M}}^{\frac{-1/2}{M}} - \int_{\frac{1/2}{M}}^{\frac{L+\frac{1}{2}}{M}} \right\} \\ & \times \tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(x)}{\mathrm{Re}\,\Lambda_j(x)} \right) \mathrm{d}x + \mathrm{O}(M^{-1}) \\ = & \left[\tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(L)}{\mathrm{Re}\,\Lambda_j(L)} \right) - \tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(-L)}{\mathrm{Re}\,\Lambda_j(-L)} \right) - \tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(0^+)}{\mathrm{Re}\,\Lambda_j(0^+)} \right) \right. \\ & \left. + \tan^{-1}\left(\frac{\mathrm{Im}\,\Lambda_j(0^-)}{\mathrm{Re}\,\Lambda_j(0^-)} \right) \right] (\mathrm{Frac}(\epsilon + \tfrac{1}{2}) - \tfrac{1}{2}) + \mathrm{O}(M^{-1}) \\ = & \, \pi k_j(\mathrm{Frac}(\epsilon + \tfrac{1}{2}) - \tfrac{1}{2}) + \mathrm{O}(L^{-1}, M^{-1}) \end{aligned}$$

where

$$k_j = \begin{cases} \theta_j + \mu_j/|\mu_j| & \text{if } \mu_j \neq 0 \\ \theta_j & \text{if } \mu_j = 0 \end{cases} \tag{15}$$

$$\theta_j = \begin{cases} \lambda_j/|\lambda_j| & \text{if } n_j > -\frac{1}{2} \text{ and } \lambda_j \neq 0 \\ (2/\pi)\tan^{-1}(\lambda_j) & \text{if } n_j = -\frac{1}{2} \\ 0 & \text{if } n_j < -\frac{1}{2} \text{ or } \lambda_j = 0 \end{cases} \tag{16}$$

and we have chosen $\tan^{-1}(\lambda_j) \in (-\pi/2, \pi/2]$. Then, the quotient (13) of the jth and $(j+1)$th determinants contributes to the imaginary part of $\Gamma(A)$ with the term

$$\pi(k_{j+1} - k_j)(-\epsilon + \mathrm{Int}(\epsilon + \tfrac{1}{2})).$$

It can be easily verified that the term $-\pi(k_{j+1} - k_j)\epsilon$ is just what we obtain in perturbation theory. Radiative corrections only pick up the Chern–Simons term of $\mathrm{Im}\,\Gamma(A)$. The term

$\pi(k_{j+1} - k_j)\mathrm{Int}(\epsilon + \frac{1}{2})$ is a purely nonperturbative effect. Collecting all the contributions of these couples of determinants we obtain

$$\mathrm{Im}\,\Gamma(A) = k_{\mathrm{eff}}\pi(-\epsilon + \mathrm{Int}(\epsilon + \tfrac{1}{2})) \tag{17}$$

where

$$k_{\mathrm{eff}} = -\theta_0 - \sum_{j=1}^{N}(-1)^j\left(\frac{\mu_j}{|\mu_j|} + \theta_j\right). \tag{18}$$

This $(0+1)$-dimensional coefficient k_{eff} equals the $(2+1)$-dimensional one in equation (1) obtained in [7]. Therefore, the ambiguities in the $(0+1)$-dimensional reflection anomaly are the same as the ambiguities in the $(2+1)$-dimensional parity anomaly.

For $N = 1$ and $\lambda_0 = \lambda_1 = 0$ we obtain the standard Pauli–Villars result $k_{\mathrm{eff}} = \pm 1$ obtained also in $2+1$ dimensions in [2] in the Abelian case ($k_{\mathrm{eff}} = \pm\frac{1}{2}$ in the non-Abelian case). It is obtained also in $2+1$ dimensions in [3–5] by using η-function regularization and in [11] by using ζ-function regularization.

For arbitrary N and $n_j \neq -\frac{1}{2}$ we obtain that k_{eff} may be any odd number. This result is also obtained in $2+1$ dimensions in [8] by using several Pauli–Villars fields and in [6] in the lattice.

If some $n_j = -\frac{1}{2}$, we obtain that k_{eff} may be any real number, even 0. $k_{\mathrm{eff}} = 0$ is also obtained in 2+1 dimensions in [9] by using an infinite number of Pauli–Villars fields. Nevertheless, regularization (11) with some $n_j = -\frac{1}{2}$ or the infinite Pauli–Villars scheme proposed in [9] are nonlocal regularizations. Then, when using these regularizations, the absence of undesirable nonphysical effects such as nonunitarity should be yet analysed before talking about absence of parity or reflection anomaly.

4. Physical origin of the reflection anomaly

In order to analyse the physical origin of the $(0+1)$-dimensional reflection anomaly and, in general, the reflection anomaly in odd dimensions, we must compare the classical and quantum symmetries of the theory.

The classical action (5) is invariant under a reflection transformation,

$$\begin{aligned} A(t) &\longrightarrow A^T(t) = -A(-t) \\ \psi &\longrightarrow \psi^T(t) = \psi(-t) \\ \psi^* &\longrightarrow \psi^{*T}(t) = -\psi^*(-t) \end{aligned}$$

and under a $U(1)$-gauge transformation $\mathrm{e}^{\mathrm{i}\phi(t)}$,

$$\begin{aligned} A &\longrightarrow A^\phi = A + \frac{\mathrm{d}\phi}{\mathrm{d}t} \\ \psi &\longrightarrow \psi^\phi = \mathrm{e}^{-\mathrm{i}\phi}\psi \\ \psi^* &\longrightarrow \psi^{*\phi} = \mathrm{e}^{\mathrm{i}\phi}\psi^*. \end{aligned}$$

Using the invariance of the action (5) under these transformations, we may deduce from the functional integral definition of $\Gamma(A)$ in (6) the formal identities

$$\mathrm{e}^{\Gamma(A^\phi)} = \mathrm{e}^{\Gamma(A)} \tag{19}$$

and

$$\mathrm{e}^{\Gamma(A^T)} = \mathrm{e}^{\Gamma(A)}. \tag{20}$$

However, $\epsilon \longrightarrow -\epsilon$ under a parity transformation and $\epsilon \longrightarrow \epsilon + n$ under a gauge transformation of winding number n. Therefore, the explicit calculations of $\Gamma(A)$ in (8) or $\mathrm{Im}\,\Gamma(A)$ in (17) show that the identity (19) is not only formal, but true. In contrast, the identity (20) is no longer valid and this fact constitutes the reflection anomaly.

Responsibility for the anomaly is often attributed to the regularization. Certainly, the true properties of the quantum theory may depend on the regularization prescription introduced to properly define the theory and may not coincide with the formal properties deduced before introducing the regularization. This is just what happens here and is argued more precisely in the following way. The quantum action defined in (6) has not a precise sense because it is divergent and therefore, the identities (19) and (20) are nothing but formal properties. The effective action is properly defined in (11) once the regularization (generalized Pauli–Villars) has been introduced (standard Pauli–Villars regularization (7) is a particular case with $N = 1$, $\mu_1 = 1$ and $\lambda_0 = \lambda_1 = 0$). Formula (11) may also be written in the form

$$\mathrm{e}^{\Gamma(A)} = \lim_{M\to\infty} \frac{\int \delta\psi^*\delta\psi \prod_{j=1}^{N} \delta\chi_j^*\delta\chi_j \, \mathrm{e}^{-S_0(A,\psi^*,\psi)-\sum_{j=1}^{N}(-1)^j S_j(A,\chi_j^*,\chi_j)}}{\int \delta\psi^*\delta\psi \prod_{j=1}^{N} \delta\chi_j^*\delta\chi_j \, \mathrm{e}^{-S_0(0,\psi^*,\psi)-\sum_{j=1}^{N}(-1)^j S_j(0,\chi_j^*,\chi_j)}}$$

where

$$S_0(A,\psi^*,\psi) = \int_{-T}^{T} \psi^*(t) \left[d_A + \mathrm{i}\lambda_0 \frac{d_A^2}{M} \left(1 + \frac{d_A^2}{M^2}\right)^{n_0} \right] \psi(t)\, \mathrm{d}t \tag{21}$$

is a generalization of the interaction $S(A,\psi^*,\psi)$ between the gauge field A and the fermion field ψ and

$$S_j(A,\chi_j^*,\chi_j) = \int_{-T}^{T} \chi_j^*(t) \left[d_A + \mathrm{i}\lambda_j \frac{d_A^2}{M} \left(1 + \frac{d_A^2}{M^2}\right)^{n_j} + \mathrm{i}\mu_j M \right] \chi_j(t)\, \mathrm{d}t \tag{22}$$

is a generalization of the classical Pauli–Villars interaction action between the gauge field and a heavy ghost fermion $\chi_j(t)$ of large mass M. Actions $S_0(A,\psi^*,\psi)$ and $S_j(A,\chi_j^*,\chi_j)$ are gauge invariant, but not reflection invariant: the terms $\mathrm{i}\mu_j M\chi_j^*\chi_j$, $\mathrm{i}\lambda_0\psi d_A^2(1 + d_A^2/M^2)^{n_0}\psi/M$ and $\mathrm{i}\lambda_j\chi_j d_A^2(1 + d_A^2/M^2)^{n_j}\chi_j/M$ spoil the formal reflection symmetry (20) of the theory.

Therefore, under this point of view, the anomaly is an ultraviolet effect: the high modes ($\pi(k+\epsilon+\frac{1}{2})/T$ for k large) of the spectrum make the determinant of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}+A(t)$ divergent and the ghost modes $\Lambda_j^{\pm1}(k + \epsilon + \frac{1}{2})$ are introduced to attenuate this ultraviolet behaviour. The price to pay is that the ghost modes spoil the reflection symmetry.

However, we can argue that the reflection anomaly is an intrinsic property of the pure $(0 + 1)$-dimensional physical theory. For that purpose we just need to study the behaviour under gauge and reflection transformations of the spectrum $\{\lambda_k(A) \equiv \pi(k+\epsilon+\frac{1}{2})/T, k \in Z\}$ of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ defining $\Gamma(A)$ in (6). This spectrum is represented in the left vertical line of figure 1 for a generic ϵ (for a generic gauge field $A(t)$).

Under a gauge transformation $\mathrm{e}^{\mathrm{i}\phi(t)}$ of winding number n, the Chern–Simons term $\epsilon(A)$ transforms into $\epsilon(A) + n$. Then, the only effect of the gauge transformation is a shift of n positions in the eigenvalues, $\{\pi(k + \epsilon + \frac{1}{2})/T, k \in Z\} \longrightarrow \{\pi(k + \epsilon + \frac{1}{2} + n)/T, k \in Z\}$ and therefore, the spectrum of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ remains invariant (the movement of the eigenvalues for the particular case of $n = 2$ is represented in figure 1). Hence, the determinant of $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ in formula (6) should be gauge invariant and in fact, the final calculation (8) is gauge invariant.

On the other hand, under a reflection transformation, $\epsilon(A) \longrightarrow -\epsilon(A)$, the spectrum transforms in the form $\{\pi(k + \epsilon + \frac{1}{2})/T, k \in Z\} \longrightarrow \{\pi(k - \epsilon + \frac{1}{2})/T, k \in Z\}$. Therefore,

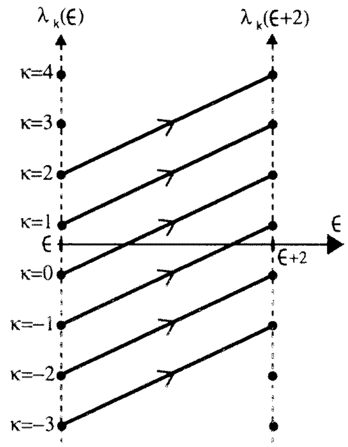

Figure 1. The spectrum of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ is shifted n positions under a gauge transformation of winding number n. This picture shows the spectrum transformation for $n = 2$.

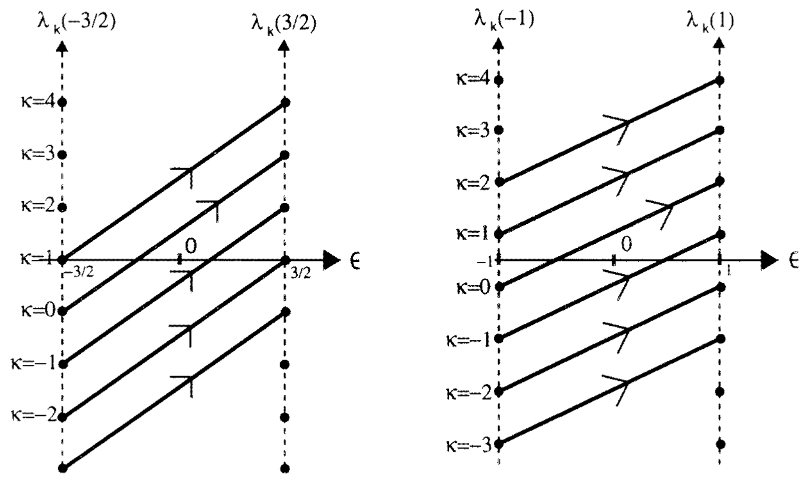

Figure 2. The spectrum of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ is reflection invariant only for integer or half-integer values of ϵ for which the spectrum is symmetric with respect to the origin. These figures represent the spectrum evolution for $\epsilon = -\frac{3}{2}$ and $\epsilon = -1$ respectively when $\epsilon \longrightarrow -\epsilon$.

the effect of the reflection transformation is a shift of the eigenvalues a quantity $2\pi\epsilon/T$. Then, the spectrum is not invariant under this transformation, unless ϵ is an integer or half-integer number (the transformation of the spectrum for the particular cases of $\epsilon = -\frac{3}{2}$ and $\epsilon = -1$ is represented in figure 2). Then, the product of these eigenvalues in formula (6) should be reflection invariant only for integer or half-integer values of ϵ . In fact, the final result (8) is invariant under the transformation $\epsilon \longrightarrow -\epsilon$ only in these cases.

The above discussion is only formal, because it is based on formula (6), that is divergent. However, for any $M > 0$, the spectrum $\{\pi(k + \epsilon + \frac{1}{2} + \mathrm{i}M)/T,\ k \in Z\}$ of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t) + \mathrm{i}M$ in equation (7) or the spectrum $\{\Lambda_j(k+\epsilon+\frac{1}{2}),\ k \in Z\}$ of the operator $d_A + \mathrm{i}\lambda_j(1 + d_A^2/M^2)^{n_j} d_A^2/M + \mathrm{i}\mu_j M$ in equation (12) are, as well as that of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$, gauge invariant and parity noninvariant, but for integer or half-integer values of ϵ. Therefore, the above discussion also holds for the regularized theory (7) or (11) and is not only formal, but rigourous.

The physical origin of the $2+1$ or higher-dimensional reflection anomaly may be described in a similar way. On the one hand, we can blame the regularization prescription†. On the other hand, we can realize that the reflection anomaly is an intrinsic property of the pure odd-dimensional physical theories. Denote by $\partial\!\!\!/_A \equiv \gamma^\mu(-\mathrm{i}\partial_\mu + A_\mu(x))$ the Euclidean $(2n+1)$-dimensional Dirac operator, where $\gamma^\mu = (\gamma^\mu)^+$ and $\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu}$. It can be trivially shown that the spectrum of the Dirac operator, $\sigma(\partial\!\!\!/_A)$, verifies $\sigma(\partial\!\!\!/_{A^\phi}) = \sigma(\partial\!\!\!/_A)$, but $\sigma(\partial\!\!\!/_{A^T}) = -\sigma(\partial\!\!\!/_A)$. Therefore, as well as in the $(0+1)$-dimensional case, the spectrum in not reflection invariant, but for special values of the gauge field for which the spectrum is symmetric with respect to the origin.

The above discussion does not hold in even dimensions. Although $\sigma(\partial\!\!\!/_{A^T}) = -\sigma(\partial\!\!\!/_A)$ also in even dimensions, the γ^5 matrix prevents the appearance of the reflection anomaly because the spectrum is symmetric with respect to the origin for any $A_\mu(x_\nu)$.

5. Conclusions

The $(0+1)$-dimensional model (5) of Dirac fermions interacting with an Abelian gauge field and defined over a circle has been quantized in the path integral formalism integrating out the fermion modes for a given background gauge field (6).

The effective action has been calculated by computing the determinant of the operator $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + A(t)$ regularized by means of the standard Pauli–Villars procedure prescribed in equation (7) and by means of the generalized scheme (11).

As well as in $2+1$ dimensions, with local regularizations, the effective coupling constant k_{eff} may be any odd number, whereas with nonlocal regularizations, k_{eff} may be any real number. Moreover, there are certain special nonlocal regularizations for which $k_{\mathrm{eff}} = 0$. If these nonlocal regularizations do not produce undesirable nonphysical effects such as nonunitarity for example, we could speak about the absence of anomaly. Otherwise, we could be dealing with a different theory, but this is still an open question [10].

Formally, the effective action (6) is gauge and reflection invariant (see equations (19) and (20) respectively). However, the regularized effective action (17) (in particular (8)), is not reflection invariant. In any odd-dimensional theory of Dirac fermions, the physical origin of the reflection anomaly may be explained in two ways. One of them is based on the regularization properties: parity-breaking terms in the regularization are responsible for the reflection symmetry breaking. The other one is not based on the regularization: the spectrum of the Dirac operator is not reflection invariant, but for special values of the gauge field for which the spectrum is symmetric with respect to the origin.

The explicit form of the $(0+1)$-dimensional nonanalytic term $h[A]$ has been obtained in equation (10). It suggests the following possibility: the $(2+1)$-dimensional term $h[A]$ may be just the same one, where ϵ must be replaced by its $(2+1)$-dimensional partner $(\mathrm{i}/2\pi)S_{\mathrm{CS}}(A)$. In fact, it may be trivially checked that $\mathrm{Int}((\mathrm{i}/2\pi)S_{\mathrm{CS}}(A) + \frac{1}{2})$ verifies all the properties enunciated in the introduction but one: 'jumps ± 1 when, for varying A, one eigenvalue of the Dirac operator vanishes'. Verification of this property and the conjecture that $h[A] = \mathrm{Int}((\mathrm{i}/2\pi)S_{\mathrm{CS}}(A) + \frac{1}{2})$ also holds in $2+1$ dimensions deserve further investigation.

† Regularizations considered in [1–9, 11] contain parity-breaking terms or break down the symmetry in some step of their construction.

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